The Measurement of Capacitance in Terms of Resistance and Frequency

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SYNOPSIS: The adaptation of a bridge circuit due to M. Wien together with apparatus and procedure is described which permits measurement of capacitance in terms of resistance and frequency with an accuracy comparable to that of the primary standards. Among its advantages over the Maxwell method commonly employed are the use of a single frequency voltage and the fact that there is no general limitation placed on the type of condenser which may be measured or on the frequency at which the measurement may be made. The method is also applicable to the determination of inductance since its unit, like that of capacitance, may be derived from the units of resistance and frequency.

INTRODUCTION

CONDENSERS are commonly measured by comparison with standard condensers of known value by means of one or another of the well-known bridge methods. The accuracy with which such measurements can be made depends upon the accuracy with which the capacitance of the standard is known.

The unit of capacitance is derivable from those of resistance and frequency and to obtain an absolute value for a standard of capacitance, some method is required for a precise determination of capacitance in terms of frequency and resistance. Of the methods which have been proposed, few yield the accuracy with which the primary standards of resistance and frequency are known and reproducible.

A generally accepted method for the absolute determination of capacitance in terms of resistance and frequency is to use a bridge, due to Maxwell,[1] employing the alternate charge and discharge of a condenser. This method has been used successfully by the Bureau of Standards,[2] which has obtained results of high accuracy. Several fundamental limitations, however, make it difficult for general use. Because of the operation of charge and discharge it is only applicable to the measurement of capacitances which are independent of frequency.[3] Practically this limits the method to the measurement of air condensers, which in large sizes are not very stable. Moreover the balance depends on the integration of successive charges and discharges of a condenser through a galvanometer and great care is required to insure that the galvanometer integrates correctly.

[1] J. Clark Maxwell, "Electricity and Magnetism," second edition, Volume 2, pp. 776–7.

[2] E. B. Rosa and N. E. Dorsey, *Bureau of Standards Bulletin*, Vol. 1, p. 153.

[3] H. L. Curtis, *Bureau of Standards Bulletin*, Vol. 6, 1910, p. 433.

The present paper describes the adaptation of a bridge circuit due to M. Wien,[4] together with apparatus and procedure, which permits a measurement of capacitance in terms of resistance and frequency with an accuracy comparable to that of the primary standards. To illustrate the possibilities of the method in practice the results of a specific determination are included. Among its advantages over Maxwell's method are the use of a single frequency voltage and the fact that there is no general limitation placed on the type of condenser which may be measured or on the frequency at which the measurement may be made.

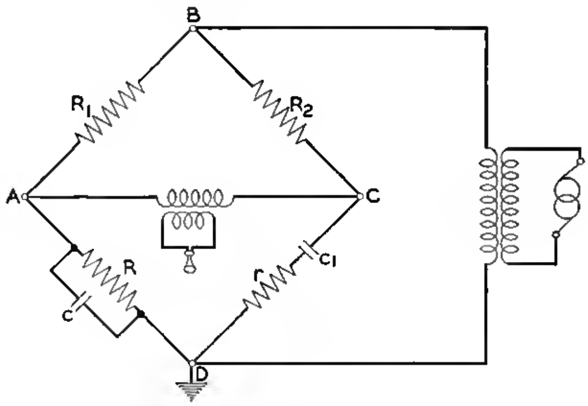

Fig. 1

The method described is also generally applicable to the determination of inductance, since its unit, like that of capacitance, may be derived from the units of resistance and frequency. The circuit and procedure to be described may be used with a change of only minor details.

In its simplest form the bridge, as shown in Fig. 1, consists of two equal resistance ratio arms, a third arm containing a capacitance and a resistance in series, and a fourth arm containing a capacitance and a resistance in parallel. A balance is easily made by varying any two of the five variables, viz., the two capacitances, the two resistances associated with them, and the frequency.

If, at balance, the frequency and any two of the other variables are known, the remaining two can be determined. Thus if the frequency, the resistance in the series arm, and the resistance in the parallel arm are known, the magnitude of both capacitances can be determined. However, the equations for balance, which are given below, are such that if the ratio of the capacitances and the value of the frequency are

[4] M. Wien, *Weid. Ann.*, 1891, p. 689.

known, the magnitudes of the capacitances can be determined from the knowledge of one only of the resistances, e.g., that in the series arm. Since the ratio of any two capacitances may be obtained with a high degree of precision by supplementary measurements, it therefore becomes possible to use the bridge just described without a knowledge of the parallel resistance, the measurement of which presents certain practical difficulties.

Although the Wien circuit is fundamentally simple, it is subject to many severe requirements when used to make an accurate determination of capacitance in terms of resistance and frequency, and must embody in its construction the refinements necessary for work of such high precision. In the Bell Telephone Laboratories there is available a capacitance bridge [5] of high precision, which is ordinarily used for the direct comparison of capacitances, and which, with slight modifications, is readily adapted to this purpose.

Theory of the Circuit

If in Fig. 1 the ratio arms are equal in resistance and phase angle, the equation of balance may be written

$$\left(\frac{1}{R} + j\omega C\right)\left(r + \frac{1}{j\omega C_1}\right) = 1,$$

and separating reals from imaginaries

$$\frac{C}{C_1} = 1 - \frac{r}{R} \tag{1}$$

and

$$CC_1 = \frac{1}{rR\omega^2}. \tag{2}$$

From these two equations it is obvious that, if the values of r, R and ω, are known, the true values of C and C_1 can be determined.

However, the method of calibrating the capacitance bridge, which is described below and which is carried out irrespective of this determination gives the value of $\frac{C}{C_1}$, precisely, and this allows the reduction of the quantities to be determined to two, ω and R or ω and r.

Let C and C_1 now be taken as the values of the two condensers as measured on the capacitance bridge to determine their ratios $\frac{C}{C_1}$.

[5] G. A. Campbell, *Elect. World and Engineer*, April 2, 1904; *Bell System Technical Journal*, July 1922; W. J. Shackelton and J. G. Ferguson, *Bell System Technical Journal*, Jan. 1928.

Since this ratio as measured is the true ratio, both of the measured values must be multiplied by the same factor to give the true values, and the following substitutions may be made in formulæ (1) and (2):

$$KC_1 \text{ for } C_1$$

and

$$KC \text{ for } C,$$

where K is the correction factor necessary to reduce the values measured on the bridge to their true values. The formulæ now become

$$\frac{C}{C_1} = 1 - \frac{r}{R}$$

and

$$K^2CC_1 = \frac{1}{rR\omega^2},$$

from which, eliminating R by the use of the ratio $\frac{C}{C_1}$

$$K = \frac{1}{rC_1\omega}\sqrt{\frac{C_1 - C}{C}}.$$

In the foregoing C and C_1 are assumed to be pure capacitances, and r and R pure resistances. Of course in practice neither pure capacitances nor pure resistances are obtainable. The former will have some slight conductance and the latter some slight reactance. If we use condensers having small losses, and resistances having small phase angles, the conductance of the condenser C (Fig. 1) may be considered as a resistance in parallel with R, and that of C_1 as a resistance, r_1, in series with r. Similarly, the reactance of R may be considered as a capacitance C', either positive or negative, in parallel with C, and the reactance of r as a capacitance, C_1', in series with C_1. Of these quantities the conductance of C may be neglected, since the use of ω, r, and the ratio $\frac{C}{C_1}$ as parameters eliminates R from the formula for K, and hence it is unnecessary to know it exactly. Including these second order quantities the formula for K becomes, using the notation above,

$$K = \frac{1}{(r + r_1)\left(\frac{C_1C_1'}{C_1 + C_1'}\right)\omega}\sqrt{\frac{\frac{C_1C_1'}{C_1 + C_1'} - (C + C')}{C + C'}}.$$

Now in the range of impedances actually used in the following determinations of K it was readily possible to obtain resistance units for r in

which the reactance was so small that $\frac{C_1 C_1'}{C_1 + C_1'}$ was no different from C_1 to the order of accuracy of the determinations. The parallel capacitance of R in the cases where single unit resistances only were used could also be made negligibly small compared with C in some cases, though the resistance values of R were in general considerably higher than those of r, and it was therefore more difficult to secure very small phase angles in the former. However, in a large number of the determinations a shielded resistance box was used for R, its phase angle was some 5 to 10 times that of the single units, and too large to neglect. Accordingly C_1' can be eliminated from the formula for K for the purpose of this investigation while C' cannot. The formula may then be written in the more simple form:

$$K = \frac{1}{(r + r_1) C_1 \omega} \sqrt{\frac{C_1 - (C + C')}{C + C'}}. \tag{3}$$

The nominal values of the first order quantities used in the actual determinations are shown in Table 1. In this table Q is the ratio of reactance to resistance of either of the total arm impedances r and C_1 or R and C.

TABLE I

Nominal Capacitance and Resistance Combinations Used in Determination of K

C_1 $\mu f.$	r ohms	C $\mu f.$	R ohms	f cycles	Q
.4	690	.1	920	1000	.6
.2	800	.1	1600	1000	1.0
.4	400	.2	800	1000	1.0
.1	1000	.072	3520	1000	1.6
.2	1000	.078	1640	1000	.8
.3	1000	.066	1280	1000	.6
.4	1000	.055	1160	1000	.4
.1	1000	.039	1630	2000	.8
.2	1000	.027	1160	2000	.4
.3	1000	.020	1070	2000	.3
.4	1000	.015	1039	2000	.2
.1	500	.091	5500	1000	3.2
.2	500	.143	1750	1000	1.6
.3	500	.158	1055	1000	1.2
.4	500	.154	810	1000	.8
.1	500	.072	1760	2000	1.6
.2	500	.078	820	2000	.8
.3	500	.066	640	2000	.6
.4	500	.055	580	2000	.4

$$K = \frac{1}{\omega C_1 r} \sqrt{\frac{C_1 - C}{C}}.$$

As mentioned above the method of this paper may obviously be extended to the measurement of inductance in terms of resistance and frequency. If in the circuit of Fig. 1, C_1 is replaced by an inductance L_1 and C by an inductance L, at balance

$$L = \frac{r^2 + \omega^2 L_1^2}{L_1 \omega^2}. \tag{4}$$

If as before L_1 is known in terms of L, i.e., if

$$L_1 = AL,$$

L_1 can be eliminated from (4) and

$$L = \frac{r}{\omega}\sqrt{\frac{1}{A(1-A)}}. \tag{5}$$

Expressing the relation (5) in terms of K, as in (3) above,

$$K = \frac{r}{\omega}\sqrt{\frac{1}{L_1(L - L_1)}}. \tag{6}$$

This formula neglects the effective resistance of L_1. The accuracy with which the value of K can be determined will in practice probably depend upon the accuracy with which the effective resistance of the inductance L_1 can be determined, which in general will be somewhat less than the accuracy of determining the conductance of the corresponding capacitance.

Description of Apparatus

The bridge equipment was a completely shielded equal-ratio bridge built for the comparison of capacitance and including standard condensers in the bridge itself. In its adaptation to the Wien circuit the standard condensers were cut out leaving a pair of equal-ratio resistance arms, properly shielded. The two additional arms were made up of external resistances and the condensers being measured.

A description of the arrangement of the standards in the capacitance bridge, however, is necessary to explain the means of obtaining the precise value of the ratio of any two capacitances. The capacitance standards, self-contained in the bridge, are variable from 0 to 1 μf and are arranged in decade form, first an air condenser with a range of slightly more than $10 \mu\mu f$, then fixed condensers up to 1 μf in 5 additional decades, each consisting of unit condensers controlled by 10 point switches. An external capacitance is measured by turning the

dials of the standard capacitance until a balance is obtained and the value is then read from the dial settings and the reading of the air condenser, which has a minimum scale division of .2 $\mu\mu f$.

The bridge condensers cannot be made exactly direct reading, and for accurate work the bridge must be calibrated. This calibration may be made very simply due to the fact that the maximum setting on any dial is approximately equal to one step on the next higher dial. By the use of an auxiliary external condenser it is possible to get a balance with any desired setting of the bridge. Thus the maximum of one dial may be compared with each individual step of the next higher dial by balancing the bridge first with the maximum setting of the lower dial and then with that dial set at zero and the next higher dial moved up one step, no change being made in the auxiliary condenser. The change in capacitance required for balance, that is, the difference between the dial settings, is read on the air condenser. Since the condensers in each decade are completely shielded from those in the other decades this procedure gives an accurate comparison of the ten steps of any dial with one another, and with the maximum setting of the next lower dial. Evidently an extension of this method will furnish a precise comparison of any bridge setting with any other, although it gives no information as to the absolute values of any of the settings.

In practice after the above "step-up" calibration, as it is called, is performed the values of all the bridge condensers are computed in terms of an assumed value of a single one. This furnishes a bridge calibration of which the consistency is dependent only on the accuracy of the "step-up" and of which the accuracy is dependent only on the value of the single calibrating standard. In general the assumed value of the calibrating standard will be in error, its true value being a constant, K, times its assumed value. Any reading on the bridge using the calibration will, therefore, require a correction by this same factor K. Now let us suppose that this bridge has exactly equal ratio arms, is calibrated as described above and is used to measure successively two capactitances whose measured values are found to be C and C_1. Their true values will then be KC and KC_1 and their ratio will be $\frac{C}{C_1}$, which is the true ratio between the capacitances irrespective of the value of K, that is, irrespective of the absolute accuracy of the measured values. By means of this type of precision capacitance bridge the ratio between any two capacitances may thus be obtained regardless of the absolute accuracy with which the capacitance of either is known.

Actually the best known value is always assumed for the capacitance used as the standard in computing the calibration of the bridge, and

accordingly the constant K by which the calibrated values of the bridge condensers must be multiplied to give their absolute values, is always very near unity. If the absolute value of the capacitance of a single condenser can be determined the factor K can readily be evaluated by measuring this known condenser on the capacitance bridge. If K is known the absolute value of any condenser can then be determined by measurement on the capacitance bridge because of the consistency of the bridge calibration. The practical reason, therefore, for determining the absolute value of a primary standard of capacitance in terms of resistance and frequency is to permit the determination of the error in the bridge calibration, i.e., to evaluate K. Accordingly, the actual measurements are carried out from the viewpoint of determining the value of K for the precision capacitance bridge rather than from the viewpoint of determining the absolute value of the capacitance of a single condenser. Of course, the latter determination is included in the former.

Special Apparatus

Aside from the shielded capacitance bridge the following special apparatus employed in the determinations is worthy of mention. The unit standard condensers were dry stack mica condensers potted in an asphalt moisture-proofing compound and shielded by brass cans. They had been kept in the laboratory a number of years so that they were thoroughly aged and their values extremely stable. The phase difference of these condensers was very small, even for high grade mica condensers.

Unit resistances were made up especially for this series of tests and consisted of bifilar windings in 100 ohm sections connected in series on hard rubber spools ¾ in. in diameter. No. 40 B. & S. gauge advance wire was used throughout. All the coils had phase angles less than .1 minute at 1,000 cycles. The 6 dial shielded resistance box used in some of the measurements was a laboratory standard variable from .01 to 10,000 ohms and calibrated for phase angle. The oscillator employed as a source of current was a specially constructed vacuum tube oscillator designed to maintain an extremely constant frequency, and to deliver a practically pure sine wave. The reference standard of frequency was a 100-cycle tuning fork surrounded by a constant temperature bath,[6] the average frequency of which from day to day was constant to .001 per cent. The reference standard of resistance against which the resistances of the units were calibrated was of the well-known

[6] J. W. Horton, N. H. Ricker, W. H. Marrison, "Frequency Measurement in Electrical Communication," *A. I. E. E. Transactions*, June, 1923.

National Bureau of Standards type,[7] calibrated by the Bureau of Standards.

Experimental Procedure

The procedure used in the determination of K, was briefly as follows: Separate unit mica condensers were selected for C and C_1. Each was measured on the capacitance bridge by itself to determine its value in terms of the bridge calibration, and in addition its series resistance was measured by comparison with the air condensers of the bridge, the series resistance of the bridge condensers being eliminated by virtue of the construction of the bridge,[5] and the method of making the measurement. At the same time the resistance of r and R, high quality resistance units, was determined by the customary Wheatstone bridge method, and their phase angles were measured by comparison with standards of which the phase angle was known to .02 minute at 1,000 cycles. The series impedance was then placed in one arm of the capacitance bridge which had previously been balanced at the frequency in question, and the parallel impedance in the other arm. The bridge was then rebalanced by varying slightly the small air condenser in the bridge and the frequency. The change in the bridge air condenser represents an algebraic addition to the capacitance C necessary because the quantities C, C_1, R, and r were not perfectly adjusted to their nominal values and because K is not exactly equal to 1. The change in frequency is necessary for the same reason. The true frequency was then determined by comparison with the laboratory standard by means of the cathode ray oscillograph.[8] For some of the determinations the bridge condensers were used for C instead of an external unit, and a shielded six dial resistance box, variable from .01 to 10,000 ohms instead of a unit resistance for R. In this case the final balance was obtained by varying the bridge condenser and R instead of the bridge condenser and the frequency. The vacuum tube oscillator used as a frequency source was capable of maintaining a frequency constant to better than .001 per cent for the duration of the tests. Sets of tests were made at three different times with an interval of about a month between them. The tests were made at frequencies of 1,000 and 2,000 cycles.

The results of the determination at each frequency are contained in Tables II and III respectively.

[7] E. B. Rosa, "A New Form of Standard Resistance," *Bulletin, Bureau of Standards*, Vol. 5, p. 413.

[8] F. J. Rasmussen, "Frequency Measurements with the Cathode Ray Oscillograph," *A. I. E. E. Journal*, January, 1927.

TABLE II

Determination of K at 1000 Cycles

Those readings made on any given day are grouped together.

C' $\mu\mu f$	$(C + C')$ μf	r_1 ohm	$r_1 + r$ ohms	C_1 μf	$C_1 - (C + C')$ μf	f cycles	K	$d \times 10^5$
+4	.100339	.15	687.08	.400130	.299791	1000.38	1.00030	+2
−3	.100047	.15	793.53	.199133	.099086	1002.13	1.00023	−5
0	.199733	.15	397.29	.400130	.200397	1002.57	1.00025	−3
+4	.100336	.15	687.03	.400144	.299808	1000.49	1.00028	0
−3	.100048	.15	793.45	.199142	.099094	1002.19	1.00028	0
0	.199725	.15	397.32	.400144	.200419	1002.61	1.00018	+10
+19	.071836	.35	998.44	.100297	.028461	1000.06	1.00031	+3
+20	.077748	.15	998.24	.199142	.121394	"	1.00036	+8
+15	.065825	.19	998.28	.301655	.235830	"	1.00032	+4
+13	.054783	.15	998.24	.400144	.345361	"	1.00036	+8
+19	.091217	.35	500.50	.100297	.009080	1000.00 ±.05	1.00032	+4
+22	.143050	.15	500.30	.199142	.056092	"	1.00031	+3
+14	.158788	.19	500.34	.301655	.142867	"	1.00024	−4
+46	.154923	.15	500.30	.400144	.246221	"	1.00023	−5
+19	.071841	.35	998.42	.100294	.028453	"	1.00023	−5
+20	.077765	.15	998.22	.199135	.121370	"	1.00025	−3
+15	.065842	.19	998.26	.301644	.235802	"	1.00019	−9
+13	.054801	.15	998.22	.400124	.345323	"	1.00029	+1
+19	.071786	.35	998.42	.100294	.028508	1001.30	1.00031	+3
+20	.077638	.15	998.22	.199131	.121493	"	1.00031	+3
+15	.065703	.18	998.25	.301641	.235938	"	1.00033	+5
+13	.054673	.15	998.22	.400122	.345449	"	1.00031	+3
+22	.142942	.15	500.35	.199131	.056189	"	1.00022	−6
+14	.158586	.18	500.30	.301641	.143055	"	1.00022	−6
+46	.154661	.15	500.35	.400122	.245461	"	1.00026	−2
+40	.100209	.15	687.03	.400122	.299913	1001.30	1.00030	+2
+19	.100134	.15	793.47	.199131	.098997	"	1.00027	−1

n = no. of observations d = deviation from mean Av. 1.00028

$\sigma = \sqrt{\frac{\Sigma d^2}{n}}$ not defined until Table 6. σ .000047

3σ .00014

Final Value of K 1.00034 ± .00003.*

To determine the effect of any inequality in the ratio arms of the bridge tests were made first with the series circuit in one arm of the bridge and then in the other at the time of each series of tests. In each case the reversal was found to cause a change of +.008 per cent in the value of K. Hence, the error in the determinations caused by all bridge inequalities was assumed as −.004 per cent; i.e., .004 per cent should be added to all determinations.

* The final value was obtained by adding .00002 for a known error in frequency and .00004 for the ratio arm error of the bridge to the average of the above determinations. The accuracy of the final value is equal to $\pm \frac{3\sigma}{\sqrt{n}}$.

TABLE III

DETERMINATION OF K AT 2,000 CYCLES

C' $\mu\mu f$	$(C + C')$ μf	r_1 ohm	$r_1 + r$ ohms	C_1 μf	$C_1 - (C + C')$ μf	f Cycles	K	$d \times 10^6$
+20	.038819	.16	998.23	.100286	.061467	2000.00	1.00025	−1
+13	.027499	.06	998.13	.199108	.171609	2000.00	1.00028	+2
+14	.019687	.08	998.15	.301597	.281910	2000.00	1.00030	+4
+14	.015274	.06	998.13	.400064	.384790	2000.00	1.00025	−1
+22	.071752	.16	500.30	.100286	.028534	2000.00	1.00021	−5
+46	.077563	.06	500.20	.199108	.121545	2000.00	1.00023	−3
+43	.065626	.08	500.22	.301597	.235971	2000.00	1.00021	−5
+34	.054604	.06	500.20	.400064	.345460	2000.00	1.00025	−1
+20	.038778	.16	998.23	.100286	.061508	2001.66	1.00030	+4
+13	.027457	.06	998.13	.199116	.171659	2001.66	1.00032	+6
+22	.071712	.16	500.33	.100286	.028574	2001.66	1.00032	+6
+46	.077475	.06	500.23	.199116	.121641	2001.66	1.00025	−1
+43	.065529	.08	500.25	.301616	.236087	2001.66	1.00027	+1
+34	.054517	.06	500.23	.400082	.345565	2001.66	1.00026	0

d = deviation from mean — Av. 1.00026

$\sigma = \sqrt{\frac{\Sigma d^2}{n}}$ — σ .000035

3σ .00010

Final value of K 1.00032 ± .00003.*

The resistance of the coils used for r was determined before and after each set of tests. Although the best grade of commercial resistance wire was used in making up the resistance coils they were found to have an appreciable temperature coefficient for work of such high precision, and due allowance had to be made for temperature variations in the computation of the results. The temperature coefficients of the resistances used are contained in Table IV.

TABLE IV

TEMPERATURE COEFFICIENTS OF RESISTANCE COILS USED AS r

Nominal Resistance of Coil	Temp. Coeff. % per ° C. at 20° C.
500 ohms	−.0016
400 "	+.0083
690 "	+.0013
800 "	+.0013
1000 "	+.0013

The temperature of the room in which all the tests except the measurement of resistance, were made, was held within ±1° C. As the condensers in the capacitance bridge and the special unit condensers

* The final value was obtained by adding .00002 for a known error in frequency and .00004 for the ratio arm error of the bridge to the average of the above determinations. The accuracy of the final value is equal to $\pm \frac{3\sigma}{\sqrt{n}}$.

were all so selected as to have negligible temperature coefficients over this range no temperature correction in the capacitance values nor in the value of K was necessary.

A well aged 3 dial condenser box which had just been calibrated at the Bureau of Standards was checked on the bridge at the time of the second series of tests. In this way a comparison of the true capacitance of the bridge condensers as determined by the Bureau of Standards method and as determined by the present method is afforded. This comparison is shown in Table V.

TABLE V

COMPARISON OF ACCURACY OF PRIMARY STANDARDS BY DETERMINATION OF K AND BY BUREAU OF STANDARDS CALIBRATION

	K for Bridge at 1000 cycles by method of this paper:— 2/27/26 to 3/19/26	K for Bridge at 1000 cycles by comparison with Bureau of Standards: 3/1/26
Average (27 determinations)	1.00034	(18 values) 1.00038
σ	.000047	.00005
3σ	.00014	.00015
$\frac{3\sigma}{\sqrt{n}}$	.000027	.000035

	K for Bridge at 2000 cycles by method of this paper:— 3/19/26
Average (14 determinations	1.00032
σ	.000035
3σ	.00010
$\frac{3\sigma}{\sqrt{n}}$	.000027

Note: K by method of this paper was determined by the following bridge condensers .1, .2, .3, .4, .04, .06, .07, .08, .09. K by comparison with the Bureau of Standards values on condenser box No. 26962 was determined by all the settings of the .01 and .1 dials of the bridge.

DISCUSSION OF RESULTS

In the discussion which follows an attempt will be made to point out the various sources of error which may creep into such a determination and to state briefly what precautions were taken to guard against them.

1. *Frequency Errors*

The primary error in the values of frequency used was to be found in a variation of the standard fork from its nominal value. The average frequency of this fork integrated over 24 hours against the Arlington time signals can be held constant to $\pm .001$ per cent. While this tells us little about the fluctuations from the mean value over short periods it is at least reasonable to assume that they will be of the same general order as the variation in the average. As a check on this

assumption the frequency of a very stable vacuum tube oscillator, specially constructed, when measured against the standard fork on the cathode ray oscillograph, can be shown not to vary with respect to the standard over a period of several hours by more than $\pm .001$ per cent. Accordingly unless the standard fork and the special oscillator always fluctuate in exactly the same manner, which is very unlikely, we may conclude that both remain constant to better than $\pm .001$ per cent. The use of the cathode ray oscillograph in checking the values of the oscillator used against the standard frequency introduces no error in the determination which is appreciable from the point of view of these tests.

2. *Modulation Errors*

Another type of error due to the source of frequency used was the masking of the true balance by oscillator harmonics affecting the detector system which consisted of the double-shielded output transformer of the bridge, a vacuum tube amplifier, a telephone receiver, and the human ear. The output characteristic of each of these elements is linear, or practically so, for small loads. Overloading of any one of them, however, results in a curved output characteristic which causes an appreciable amount of modulation. This is especially true of the vacuum tube amplifier. Now in a measurement of this type the bridge is balanced for the fundamental only, since the equations of balance contain the frequency as a parameter, and any harmonics present in the input will pass through into the detector circuit practically unattenuated. If they are appreciable in magnitude the elements of the detector, particularly the amplifier, become overloaded, more harmonics are generated, these harmonics are modulated in a succeeding element of the detector, and the fundamental may appear as a modulation product even though the bridge is actually balanced. Under such conditions the fundamental tone in the receiver can be eliminated only by unbalancing the bridge slightly. It was found during the tests that if the output of the oscillator was kept small no appreciable overloading occurred in the detector circuit, but that as the oscillator output was increased the amplitude of the harmonics also increased, the detector gradually became overloaded, and the bridge balance began to change as the oscillator output was varied. This effect was eliminated by the use of a filter to keep the harmonics from the oscillator out of the detector circuit.

3. *Resistance Errors*

Errors in the determination of K due to uncertainty in resistance values arose in four ways. The first source of error due to the resistance coils lay in their temperature coefficient. This coefficient was found to

be large enough to cause serious error in some of the values, one coil in particular having an extremely large temperature variation for resistances of this type. By determining the temperature coefficient of the coils used and making appropriate temperature corrections it was possible to reduce the uncertainty due to this cause to less than .001 per cent. The second source of error in resistance values resulted from the effect of humidity on the coils. They were impregnated with shellac, which absorbed moisture and swelled sufficiently to cause changes in resistance with humidity large enough to effect the results seriously. By measuring the coils both before and after each series of tests it was possible to keep the error due to this cause below .001 per cent. The use of potted coils would undoubtedly eliminate this difficulty completely. The third resistance error present arose from uncertainties as to the series resistance of the condensers C_1. These values were determined in terms of the bridge air condensers by the step-up method, on the assumption that the air condensers in the bridge have no conductance, it being eliminated by the method of measurement and by virtue of the special construction of the bridge.

In the fourth place the accuracy with which our primary standards of resistance are calibrated by the Bureau of Standards must be considered. These calibrations have been found consistent from year to year to about $\pm .001$ per cent., and accordingly can probably be relied on to that value in the future. The results furnished by the Bureau are based on their primary standards, which agree with the primary standards of leading European nations to better than $\pm .002$ per cent, hence there is little likelihood of their changing their values by the latter amount.

4. *Capacitance Errors*

The accuracy with which the ratio of any two capacitances may be determined in such an investigation is dependent upon the consistency of the bridge calibration by the step-up method. A detailed discussion of the consistency of this calibration is beyond the scope of this paper. As an indication of the order of the precision with which it is possible to obtain a comparison between two condensers by this method, the results of measurements on a standard condenser box on three different bridges all calibrated by means of the step-up are shown in Table VI. The value of $\pm 3\sigma$, which is taken as the measure of the accuracy with which measurements can be reproduced, is $\pm .004$ per cent as determined from 54 individual measurements. On this basis the error in the ratio of two condensers compared by this method will be less than $\pm .0056$ per cent ($\sqrt{2} \times .004$). Actually the error in the comparison of the values of two condensers by measurement on such a calibrated

TABLE VI

Agreement between Measurements on a Standard Condenser Box on Three Different Bridges, All Calibrated by the Step-up Method, Using the Same Primary Standard

Setting μf	No. 2 Bridge		No. 8 Bridge		No. 9 Bridge	
	d	$d^2 \times 10^8$	d	$d^2 \times 10^8$	d	$d^2 \times 10^8$
.01	−.0013	169	+.0007	49	+.0007	49
2	+.0003	9	+.0013	169	−.0017	289
3	+.0017	289	+.0017	289	−.0033	1089
4	.0000	0	+.0010	100	−.0010	100
5	+.0003	9	−.0007	49	+.0003	9
6	+.0014	196	−.0006	36	−.0006	36
7	+.0010	100	.0000	0	−.0010	100
8	+.0010	100	.0000	0	−.0010	100
9	+.0010	100	−.0010	100	.0000	0
.1	+.0013	169	+.0003	9	−.0017	289
2	+.0023	529	.0000	0	−.0021	442
3	+.0033	1089	−.0017	289	−.0017	289
4	+.0028	784	−.0012	144	−.0017	289
5	+.0030	900	−.0010	100	−.0020	400
6	+.0010	100	.0000	0	−.0010	100
7	+.0013	169	−.0007	49	−.0007	49
8	+.0013	169	+.0003	9	−.0017	289
9	+.0020	400	−.0005	25	−.0010	100

$$\Sigma d^2 \times 10^8 = 11215,$$
$$\sigma = \sqrt{\frac{11215}{54}} \times 10^{-4},$$
$$\sigma = .0014\%,$$
$$3\sigma = .0042\%.$$

Note: d = per cent deviation from the mean value for any setting. $\sigma = \sqrt{\frac{\Sigma d^2}{n}}$, where n = the number of observations.

bridge was probably appreciably less than ±.005 per cent in all cases, as the values of Table VI were obtained in the course of the routine calibration of the several bridges, while in the determinations of K only the one bridge was used and special precautions were taken to make the consistency of the step-up as high as possible. In any event this value of $\pm 3\sigma$ is appreciably less than that for a single determination of K, which ranges from ±.010 per cent to ±.014 per cent, as shown in Tables II and III. Accordingly the assumption that the consistency of the step-up method of calibration is sufficiently high for the purpose is well founded.

The accuracy of the capacitance values is also dependent upon the accuracy of the determination of the equivalent shunt capacitance of the resistance R. The phase angle of the series resistance r was

small enough to be neglected in all cases. The final determination of the shunt capacitance of R was accurate to $\pm 1 \mu\mu f$, and hence in most of the cases under consideration the resulting uncertainty in K was less than $\pm .001$ per cent.

4. *Bridge Errors*

The principal source of error in the bridge itself lies in the inequality of the ratio arms, both in magnitude and angle. Since this inequality may result from sources other than the ratio arms proper, it is best to

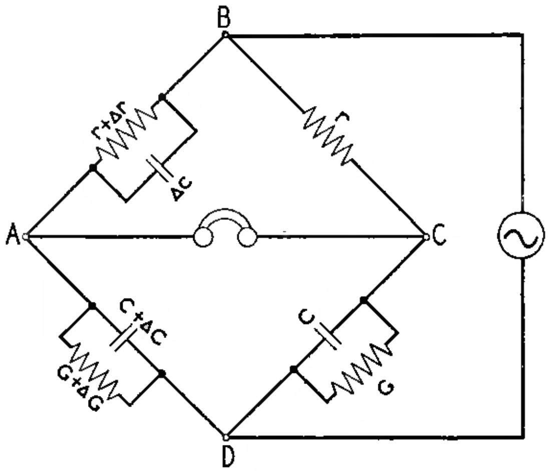

FIG. 2

ascertain it by interchanging the impedances being measured rather than by reversing the arms themselves. The following formulæ (see Fig. 2) show the errors in conductance and capacitance which may arise from ratio arm inequalities.

$$\frac{\Delta G}{2G} = -\frac{\Delta r}{r} - \frac{\omega^2 C \Delta c r}{G}, \tag{4}$$

$$\frac{\Delta C}{2C} = -\frac{\Delta r}{r} + \frac{\Delta c r G}{C}. \tag{5}$$

In the above,

G = the conductance of the unknown.

ΔG = the change in conductance due to reversing the unknown and the standard arms.

C = the capacitance of the unknown.

ΔC = the change in capacitance due to reversing the unknown and the standard arms.

r = the resistance of the ratio arms.

Δr = the resistance unbalance of the ratio arms.

Δc = the capacitance unbalance in the ratio arms.

These formulæ are not rigorous, as second order quantities have been neglected, but they are accurate to a close approximation provided the ratio arm capacitance is very small, the frequency is in the audible range, and the ratio of susceptance to conductance in the unknown is one or larger. All of these conditions obtain in the case in point.

By measuring direct and reversed an admittance having a Q (ratio of susceptance to conductance) of approximately 1 and solving the two equations simultaneously we may ascertain errors due to the differences in resistance and reactance of the ratio arms. The total change in capacitance of an admittance under test due to the resistance error of the ratio arms was found by the above method to be approximately .004 per cent; the capacitance error due to the reactance unbalance of the ratio arms was found to be $\frac{.004\%}{Q}$. Thus the combined error in capacitance due to both types of unbalance is a function of the Q of the impedance being measured. In the case of the particular type of tests being made the combined error takes the form of an error in the capacitance C, i.e., the capacitance in the shunt circuit. Table I contains a column showing the Q's of the impedances used for the tests, which range between .2 and 3.2. The corresponding error in C due to the total ratio arm unbalance varies between .014 per cent and .002 per cent (the error in C is obviously ½ of the total capacitance change resulting from the impedance arm reversal). It can easily be shown, however, from the relation between the capacitances C_1 and C of Table I that the error in K resulting from the foregoing capacitance error will in general lie between limits of .003 per cent to .005 per cent except for one or two extreme cases for which the limits are .002 per cent and .008 per cent. Accordingly the total correction due to the bridge errors was lumped at .004 as noted under "Experimental Procedure," since the few cases for which the error reaches the extreme limits are those for which the accuracy of the test as a whole is a minimum, aside from this particular type of error.

Final Accuracy of Result

The standard deviation σ for the individual determinations of K has been worked out for the values in Tables II and III. The value of σ is significant in that, provided the distribution of errors is approxi-

mately normal, over 99 per cent of all determinations will fall within limits of $\pm 3\sigma$ from the mean. In this discussion the accuracy of any measurement (exclusive of known consistent errors) will be defined as $\pm 3\sigma$. The standard deviation of the mean is given by the expression $\frac{\sigma}{\sqrt{n}}$, where n is the total number of observations. If the curve of errors is approximately normal (and we have no reason to assume otherwise), the error in the determination of K is given by $\pm \frac{3\sigma}{\sqrt{n}}$. In the absence of any systematic errors, of which none have been detected of magnitude comparable with the final accuracy of the result, the limits of accuracy are therefore, from Table V, $\pm .003$ per cent.

This limit was not exceeded in practice as is shown by the values for K at 1,000 and at 2,000 cycles in Table V. The difference between the two values of K is .002 per cent. Since the calibrations at both frequencies are based on the same original standard, namely, the 1,000-cycle value of the bridge $.01\mu f$ air condenser, and the latter is assumed not to vary with frequency over the audio range, the final result for K in the two cases should be the same within the limits of accuracy of the result.

Table V contains a comparison of the values of K as determined by the method of this report and by comparing the Bureau of Standards calibrated values on a standard condenser box with the calibration of the capacitance bridge at 1,000 cycles. The agreement between these values, .004 per cent, is very close, in view of the accuracy which the Bureau certifies and the precision to which their results are given. Although the Bureau calibration is certified only to $\pm .1$ per cent, the values are furnished to 5 significant figures and are apparently consistent to $\pm .01$ per cent or better.

It will be noted that the values of capacitance chosen for these tests were all between .01 and $.5\mu f$. It is advisable that they be kept within these limits at the frequencies used in order that the resistance values required in the determination may be easily secured and easily capable of measurement with the required precision, and that errors in capacitance due to slight changes in the position of leads and units may not be appreciable.